# Equivalence and Relational Thinking:

## Opportunities for Professional Learning

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Colleen Vale makes the case for professional learning teams collaborating together to improve their teaching and hence children's achievement. In this article she describes how this may be done. Along the way the teachers explored the idea of equivalence and the common conceptions and misconceptions held by children in their classes.

Various studies confirm that schools that succeed in improving students' mathematics learning provide regular scheduled time for teachers to collaborate for professional learning and to prepare for teaching. In these schools, collaborative teacher professional learning teams investigate children's mathematical thinking and achievement and they also review and rehearse enactment of teaching approaches (Cobb & Jackson, 2011; Kazemi & Franke, 2004). By working together to analyse students' thinking using samples of students' work, teachers can deepen their knowledge of mathematics and of their students. Likewise collaborative exploration and rehearsal of teaching approaches and learning tasks develop teachers' capacity to anticipate and scaffold students' thinking.

In this article a report on a professional learning workshop conducted for primary mathematics specialist teachers in Victoria on equivalence illustrates the way in which professional learning teams might collaborate to investigate their students' thinking and preview approaches to teaching. The teachers who participated in this workshop were experienced and taught classes from Prep to Year 6 at the time of the workshop. During the workshop, teachers were encouraged to compare and contrast students' mathematical thinking with regard to efficiency. They also trialled a 'true–false number talk' (Chapin, O'Connor & Anderson, 2009), a teaching approach and task designed to develop children's relational thinking.

## Equivalence

Equivalence is a big idea in mathematics. It describes a special relationship between mathematical objects, where these objects could be numbers, measurements, shapes, number statements or functions. Equivalence means 'is the same as.' So equivalent numbers have the same value but a different name, for example equivalent fractions; equivalent measurements are the same size but a different shape, for example the same capacity of water in two differently shaped glasses.

When children understand and use equivalence, they are able to make connections between what might otherwise seem to be quite separate mathematical ideas and procedures. They pay attention to relationships between numbers, measurements or shapes, and they use equivalence to derive mental strategies for computing operations and to solve problems (Carpenter, Franke & Levi, 2003). This approach to thinking mathematically is called *relational thinking*.

The professional learning workshop focussed on the following questions: How do teachers know when children understand equivalence and how can teachers recognise children's use of relational thinking? How do children develop the capacity for relational thinking? How can teachers scaffold this development?

## Investigating students' thinking

The primary mathematics specialist teachers brought samples of student work to the workshop and during the workshop, they collaborated to analyse and categorise written records of students' thinking. They identified misconceptions and three main thinking strategies used by their students.

### Collecting student work

Prior to the workshop the teachers asked the students in their class to solve a missing number problem. They selected one problem from the list included in Figure 1.

What is the missing number?

$$9 + 3 = \square + 7$$
$$7 + 21 = \square + 11$$
$$17 + 24 = \square + 21$$
$$45 + 37 = \square + 22$$

Figure 1. Missing number problems.

Teachers provided a variety of materials such as counters and base 10 materials so that children could choose whatever strategy they wanted to solve the problem. The teachers asked the children to record their strategy including their mental strategies or strategies using materials. Teachers were encouraged to probe children's thinking and record this thinking, especially for students who did not record a clear explanation. Teachers were asked to bring a diverse selection of students' work to the workshop.

### Analysing student work

In the workshop, teachers were organised into groups according to the missing number problem that their students had solved. They compared and contrasted children's thinking recorded in the work samples following the instructions shown in Figure 2. They were encouraged to focus on the strategies children used to find the missing number as well as their methods of calculation. They grouped students' responses and observed the number of student responses that were misconceptions or used a particular strategy.

Compare and contrast students' thinking about the same problem for the samples of student work in your group:

1. *Sort these samples into two groups:*
   - Those that demonstrate knowledge of equivalence and those that do not. How do you know?
2. *Compare students' strategies and sort samples according to strategy.*
   - Analyse only those with understanding of equivalence.
   - What different strategies did students use?
3. *Order these strategies from least efficient to most efficient.*
   - What criteria did you use?

Figure 2. Process for analysing of students' work on a missing number problem.

## Misconceptions

In general, the teachers were surprised by the high proportion of students who did not demonstrate an understanding of equivalence. Prevalence of misconceptions was higher for younger children but also evident for children in the middle and upper grades. Consistent with findings reported by researchers, the most common misconception was that equals means 'find the answer' (Carpenter et al., 2003). These children typically found the sum of numbers on the left hand side (LHS) or the sum of all numbers in the equation.

Teachers were concerned about students who appeared to have an understanding of equivalence but who made computational errors. In this workshop, the teachers were encouraged to focus their analysis on the strategies and mathematical thinking of students who were successful. Understanding the strategies used by students who successfully solved the problem enables teachers to appreciate the mathematical thinking children demonstrate. This demonstrated thinking can then be used by teachers in a carefully orchestrated whole-class discussion, thus enabling children to learn from each other (Stein, Engle, Smith & Hughes, 2010).

## Successful strategies

The teachers identified three main strategies used by students.

**Balance strategy**

These students found the sum of numbers on the left-hand side (LHS) and then used various addition strategies to find the missing number so that the sum on the right-hand side (RHS) is the same. For example:

LHS: $7 + 21 = 28$;
RHS: $\square + 11 = 28$;
17 works; $17 + 11 = 28$

**Transformation strategy**

These students found the sum on the left-hand side and then used the inverse operation, in this case, subtraction, to find the missing number. For example:

LHS: $7 + 21 = 28$; $\square = 28 - 11 = 17$

**Relational thinking**

These students looked for relationships between numbers on the opposite sides of the equals sign and used this relationship to 'balance' the sums on the left-hand and right-hand sides. For example, 11 is 10 less than 21 so add 10 to 7 to balance it up. This relationship is illustrated in Figure 3 and modelled using base 10 materials in Figures 4. Partitioning and the associative law are used in the direct modelling of the problem shown in Figure 4. This reasoning can be shown using mathematics symbols:

$$\begin{aligned} 7 + 21 &= 7 + (10 + 11) \\ &= (7 + 10) + 11 \\ &= 17 + 11 \end{aligned}$$

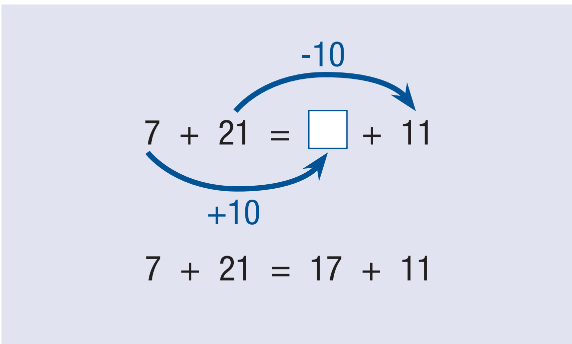


Figure 3. Relational thinking.

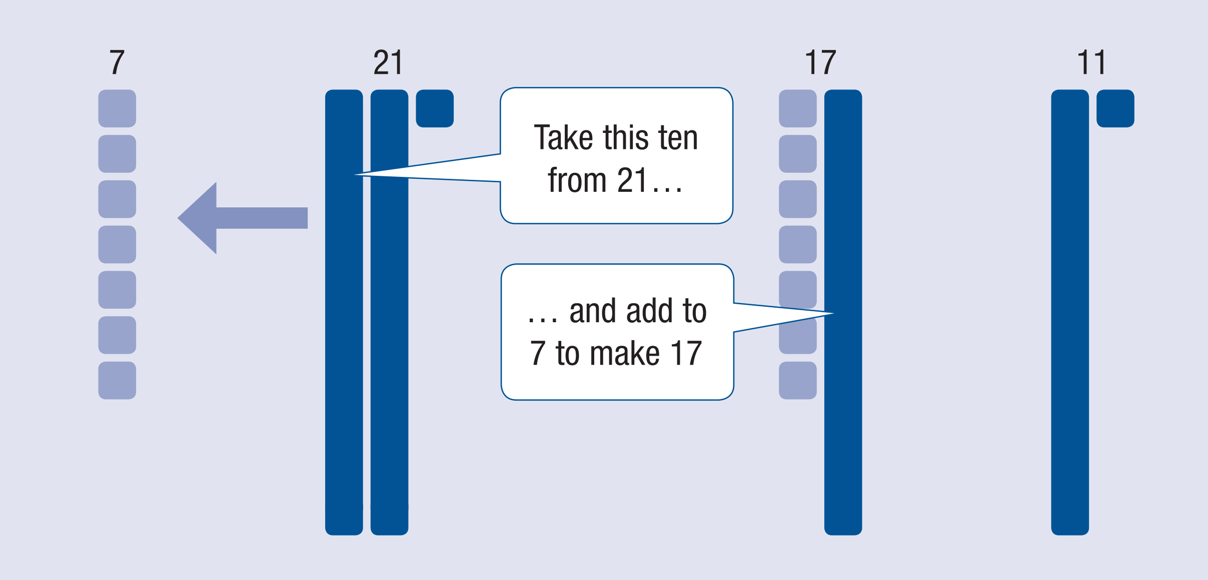


Figure 4. Relational thinking using base 10 materials.

Teachers observed that children generally used the balance or transformation strategy for each of the missing number problems. Some children used relational thinking. The teachers agreed that this was the most efficient strategy since calculation of sums on the left-hand or right-hand sides was not needed. Examples of relational thinking were found for each number problem, except for the problem, $9 + 3 = \square + 7$. In this instance, the most efficient strategy was the use of a known fact to balance the sums on each side of the equal sign.

## Computational methods

Teachers noted that children used a variety of computational methods. These included:

- direct modelling with materials or drawings using ones, or tens and ones;
- counting on, counting up to, or counting back;
- written place value algorithms for addition and/or subtraction, including the place value algorithm for addition where children used guess and check to find the missing addend;
- known facts, derived facts or mental computation.

Some students used mental computation. For example, one student used a compensation method to calculate 45 + 37 and recorded their thinking as:

45 is near 40 and 37 is near 40
$40 + 40 = 80$
$5 - 3 = 2$
$80 + 2 = 82$

When teachers analysed this student's thinking they realised that this student had used an understanding of equivalence and the associative law, possibly unconsciously, to find the left-hand side sum.

$$\begin{aligned} 45 + 37 &= (45 - 5 + 5) + (37 + 3 - 3) \text{ since } 45 = 45 - 5 + 5 \text{ and } 37 = 37 + 3 - 3 \\ &= 40 + 5 + 40 - 3 \\ &= 40 + 40 + (5 - 3) \\ &= 82 \end{aligned}$$

This example illustrates the way in which fluency with mental computation demonstrates understanding of equivalence. Teachers can therefore build on this fluency to develop students' relational thinking.

Teachers were encouraged to record examples of their students' strategies using a grid of problem solving and computational strategies (Figure 5). Teachers might choose to record children's names in the relevant cells.

| Solution strategy | METHOD OF CALCULATION | | | |
|---|---|---|---|---|
| | Direct modelling (by ones or tens) | Counting (by ones or tens) | Written algorithm | Known facts, derived facts & mental strategies |
| **Misconceptions** Equals (=) means find the answer | | | | |
| **Balance strategy** | | | | |
| **Transformation strategy** | | | | |
| **Relational thinking** | | | | |

Figure 5. Grid for recording students' strategies.

## Investigating teaching approaches

Teacher collaborative learning should also include investigation of teaching approaches (Cobb & Jackson, 2011). Many schools make use of observation of teaching in person or via video or digital recordings. Cobb and Jackson (2011) and Lampert and colleagues (2010) recommend teachers rehearse key parts of a planned lesson with colleagues. During the workshop, teachers were able to rehearse one approach for developing students' relational thinking: a true–false number talk. Other tasks were briefly described. Teachers collaborating in professional learning teams could trial and rehearse these tasks when planning lessons to address misconceptions regarding equivalence and to develop relational thinking.

### Developing understanding of equivalence

Given the high proportion of students with misconceptions regarding the meaning of the equals sign, teachers in the workshop were keen to identify tasks/approaches that would enable children to develop this understanding or confront their misconceptions. The following relevant tasks were briefly described:

- open-ended tasks such as number sentences with the operation recorded on the right-hand side of the equal sign, for example, 36 = ;
- closed problems such as missing number sentences with the operation on the right-hand side, for example, $12 = \boxed{\phantom{0}} + 7$;
- equal addition (or subtraction) cards (Stephens & Armanto, 2010);
- keeping the sum (or difference) the same (Department of Education and Early Childhood Development [DEECD], 2006).

### Developing relational thinking

To develop capacity for relational thinking, students need to collaborate through group investigations and group or whole class mathematical discussions where strategies are shared and reasoning explained. Examples include:

- equal addition (or subtraction) arrays (Stephens & Armato, 2010);
- number sentences with two unknowns (Stephens & Wu, 2009);
- mental computation problems; and
- true/false scenarios (Carpenter et al., 2003).

A true/false number talk that challenged teachers' thinking was rehearsed in the workshop. Number talks are short discussions conducted with the whole class or a small group. Students can learn from each other when teachers use targeted number talks and carefully structured share time where students present ideas and strategies and explain their thinking (Chapin et al., 2009; Stein et al., 2010). Number talks provide students with the opportunity to clarify their own thinking, consider and test other strategies, investigate and apply mathematical relationships, build a set of efficient strategies and make decisions about choosing efficient strategies. The teacher's role is to establish respectful and supportive classroom norms, ensure equitable participation and facilitate students' thinking. Effective teachers restate a student's response (revoice), call on another student to repeat this idea in their own words (repeat), ask a student to apply their reasoning to someone else's reasoning (reason) or ask another student to add on to the previous response or contribute a new idea (add on) and use wait time (Chapin et al., 2009). The teacher records the students' thinking on the whiteboard using the representation described by the student or invites the student(s) to do this.

### True/false scenario

For this particular kind of number talk the teacher chooses a mathematical statement and poses the question: "Is this true or false?" The teacher then conducts the discussion using the actions described above. True/false scenarios are useful for a range of mathematics topics as they require students to justify their thinking (Carpenter et al., 2003). They are especially useful for confronting misconceptions and proving relationships. In choosing the statement, the teacher needs to have a clear objective in mind. Some examples for equivalence and relational thinking are included in Figure 6. Using larger numbers discourages calculation and encourages relational thinking. Teachers can also plan to use a string or series of true/false scenarios. The string in Figure 6 aims to develop understanding of equivalence, the commutative property and relational thinking.

| | |
|---|---|
| $37 + 54 = 35 + 5$ True or false? | a. $3 + 5 = 8$ |
| $471 + 377 = 472 + 378$ True or false? | b. $8 = 5 + 3$ |
| $564 + 56 - 59 = 561$ True or false? | c. $8 = 8$ |
| $31 - 7 = 21 - 17$ True or false? | d. $3 + 5 = 3 + 5$ |
| | e. $3 + 5 = 5 + 3$ |
| | f. $3 + 5 = 4 + 4$ |
| | (Carpenter et al., 2003, p. 16) |

Figure 6. True/false scenarios for a number talk

## Conclusion

Investigating students' thinking and teaching practices with colleagues in your school will probably take up a few sessions but it will be worth it. Exploring equivalence provides opportunity for teachers to make further connections between mathematics topics and deepen their understanding of learning trajectories. Investigating, planning, enacting and reflecting on number talks will enable teachers to bring mathematical reasoning out into the open and enable students to learn from each other.

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